Normally preordered spaces and utilities*

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Abstract

In applications it is useful to know whether a topological preordered space is normally preordered. It is proved that every k_ω-space equipped with a closed preorder is a normally preordered space. Furthermore, it is proved that second countable regularly preordered spaces are perfectly normally preordered and admit a countable utility representation.

1 Introduction

In applications such as dynamical systems [1], general relativity [27] or microeconomics [6] it is useful to know if a topological preordered space, usually a topological manifold, is normally preordered.[1] The preorder arises from the orbit dynamics of the dynamical system; from the causal preorder of the spacetime manifold; or from the preferences of the agent in microeconomics. The condition of preorder normality can be regarded as just one first step in order to prove that the space is quasi-uniformizable or even quasi-pseudometrizable in such a way that it admits order completions and order compactifications.

The case of a preorder is often as important as the case of an order. Indeed, dynamical systems are especially interesting in the presence of dynamical cycles, and, analogously, spacetimes are particularly interesting in presence of causality violations. Also the case of a preorder is the usual one considered in microeconomics as an agent may be indifferent with respect to two possibilities in the space of alternatives (prospect space) (actually the agent can even be unable to compare them, this possibility is called indecisiveness or incomparability [2]).

It is well known that a topological space equipped with a closed order is Hausdorff. The removal of the antisymmetry condition for the order suggests to remove the Hausdorff condition for the topology. Indeed, quite often in applications, the preorder is so tightly linked with the topology that one has that two points which are indistinguishable according to the preorder (i.e. $x \le y$ and $y \le x$) are also indistinguishable according to the topology, so that even

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[1]Domain theory [12] has applications to computer science and is related in a natural way to ordered topological spaces. In this field a topological space equipped with a closed order is called *pospace* and a normally ordered space is called *monotone normal pospace*.

imposing the T_0 property could be too strong. This fact is not completely appreciated in the literature on topological preordered spaces. Nachbin, in his foundational book [29], uses at some crucial step the Hausdorff condition implied by the closed order assumption [29, Theor. 4, Chap. I]. In this work we shall remove altogether the Hausdorff assumption on the topology and in fact even the T_0 assumption. The idea is that the separability conditions for the topology should preferably come from their preorder versions and should not be added to the assumptions.

So far the only result which allows us to infer that a topological preordered space is normally preordered is Nachbin's theorem [29, Theor. 4, Chap. I], which states that a compact space equipped with a closed order is normally ordered. There are other results of this type [10, Theor. 4.9] but they assume the totality of the order. Our main objective is to prove a result that holds at least for topological manifolds and in the preordered case so as to be used in the mentioned applications. Indeed, we shall prove that the k_ω-spaces equipped with a closed preorder are normally preordered. Since topological manifolds are second countable and locally compact and these properties imply the k_ω-space property, the theorem will achieve our goal.

1.1 Topological preliminaries

Since in this work we do not assume Hausdorffness, it is necessary to clarify that in our terminology a topological space is *locally compact* if every point admits a compact neighborhood.

A topological space E is a *k-space* if $O \subset E$ is open if and only if, for every compact set $K \subset E$, $O \cap K$ is open in K. We remark that we are using here the definition given in [32], thus we do not include Hausdorffness in the definition as done in [8, Cor. 3.3.19]. Using our definition of local compactness it is not difficult to prove that every first countable or locally compact space is a k-space (modifying slightly the proof in [32, Theor. 43.9]).

A related notion is that of *k_ω-space* which can be characterized through the following property [11]: there is a countable sequence K_i of compact sets such that $\bigcup_{i=1}^{\infty} K_i = E$ and for every $O \subset E$, O is open if and only if $O \cap K_i$ is open in K_i with the induced topology (again, here E is not required to be Hausdorff). The sequence K_i is called *admissible*. By replacing $K_i \to \bigcup_{j=1}^{i} K_j$ one checks that it is possible to assume $K_i \subset K_{i+1}$; also the replacement $K_i \to K \cup K_i$ shows that in the admissible sequence K_1 can be chosen to be any compact set. We have the chain of implications: compact $\Rightarrow$ k_ω-space $\Rightarrow$ σ-compact $\Rightarrow$ Lindelöf, and the fact that local compactness makes the last three properties coincide.

We shall be interested on the behavior of the k_ω-space condition under quotient maps. Remarkably, the next proof shows that the Hausdorff condition in Morita's theorem [28, Lemmas 1-4] can be dropped.

Theorem 1.1. *Every σ-compact locally compact (locally compact) space is a k_ω-space (resp. k-space). The quotient of a k_ω-space (k-space) is a k_ω-space*

(resp. k-space). Every k_ω-space (k-space) is the quotient of a σ-compact (resp. paracompact) locally compact space.

Proof. The first statement has been already mentioned.

Let $K_\alpha, \alpha \in \Omega$, be an admissible sequence (resp. the family of all the compact sets) in E. Let $\pi : E \to \tilde{E}$ be a quotient map and let $\tilde{K}_\alpha = \pi(K_\alpha)$; then the sets $\tilde{K}_\alpha, \alpha \in \Omega$, form a countable family of compact sets (resp. a subfamily of the family of all the compact sets). Suppose that $C \subset \tilde{E}$ is such that $C \cap \tilde{K}_\alpha$ is closed in $\tilde{K}_\alpha$. We have $\pi^{-1}(C \cap \tilde{K}_\alpha) = \pi^{-1}(C) \cap \pi^{-1}(\tilde{K}_\alpha)$; thus $\pi^{-1}(C) \cap K_\alpha = \pi^{-1}(C) \cap \pi^{-1}(\tilde{K}_\alpha) \cap K_\alpha = \pi^{-1}(C \cap \tilde{K}_\alpha) \cap K_\alpha$. Let C_α be a closed set on $\tilde{E}$ such that $C_\alpha \cap \tilde{K}_\alpha = C \cap \tilde{K}_\alpha$; then $\pi^{-1}(C) \cap K_\alpha = \pi^{-1}(C_\alpha \cap \tilde{K}_\alpha) \cap K_\alpha = \pi^{-1}(C_\alpha) \cap \pi^{-1}(\tilde{K}_\alpha) \cap K_\alpha = \pi^{-1}(C_\alpha) \cap K_\alpha$. As the set on the right-hand side is closed in K_α for every α we get that $\pi^{-1}(C)$ is closed and hence C is closed by the definition of quotient topology. (Note that $\tilde{K}_\alpha$ is an admissible sequence in the k_ω-space case.)

Let K_α, $\alpha \in \Omega$, be an admissible sequence (resp. the family of all the compact sets) in E. Let $\tilde{K}_\alpha = \{(x, \alpha), x \in K_\alpha\}$, $E' = \cup_\alpha \tilde{K}_\alpha$ be the disjoint union [32] and let $g : E' \to E$ be the map given by $g((x, \alpha)) = x$ so that $\varphi_\alpha := g|_{\tilde{K}_\alpha} : \tilde{K}_\alpha \to K_\alpha$ is a homeomorphism. Let $C \subset E$ be such that $g^{-1}(C)$ is closed then for each α, $g^{-1}(C) \cap \tilde{K}_\alpha$ is closed in $\tilde{K}_\alpha$ thus $g(g^{-1}(C) \cap \tilde{K}_\alpha) = \varphi_\alpha(g^{-1}(C) \cap \tilde{K}_\alpha) = C \cap K_\alpha$ is closed in K_α because φ_α is a homeomorphism. Thus C is closed and hence g is a quotient map. It is trivial to check that E' is σ-compact (resp. paracompact) and locally compact. □

1.2 Order theoretical preliminaries

For a topological preordered space $(E, \mathscr{T}, \le)$ our terminology and notation follow Nachbin [29]. Thus by *preorder* we mean a reflexive and transitive relation. A preorder is an *order* if it is antisymmetric. With $i(x) = \{y : x \le y\}$ and $d(x) = \{y : y \le x\}$ we denote the increasing and decreasing hulls. A topological preordered space is *semiclosed preordered* if $i(x)$ and $d(x)$ are closed for every $x \in E$, and it is *closed preordered* if the graph of the preorder $G(\le) = \{(x, y) : x \le y\}$ is closed. A subset $S \subset E$, is called *increasing* if $i(S) = S$ and *decreasing* if $d(S) = S$. $I(S)$ denotes the smallest closed increasing set containing S, and $D(S)$ denotes the smallest closed decreasing set containing S. It is understood that the set inclusion is reflexive, $X \subset X$.

A topological preordered space is a *normally preordered space* if it is semiclosed preordered and for every closed decreasing set A and closed increasing set B which are disjoint, $A \cap B = \emptyset$, it is possible to find an open decreasing set U and an open increasing set V which separate them, namely $A \subset U$, $B \subset V$, and $U \cap V = \emptyset$.

A *regularly preordered space* is a semiclosed preordered space such that, if $x \notin B$ with B a closed increasing set, there is an open decreasing set $U \ni x$ and an open increasing set $V \supset B$, such that $U \cap V = \emptyset$, and analogously the dual property must hold for $y \notin A$ with A a closed decreasing set.

We have the implications: normally preordered space $\Rightarrow$ regularly preordered space $\Rightarrow$ closed preordered space $\Rightarrow$ semiclosed preordered space. A topological preordered space is *convex* [29] if for every $x \in E$, and open set $O \ni x$, there are an open decreasing set U and an open increasing set V such that $x \in U \cap V \subset O$.

2 Preorders on compact spaces and k_ω-spaces

We are interested in establishing in which way compactness and countability assumptions improve the preorder separability properties of a topological preordered space. The example below shows that these conditions do not promote semiclosed preordered spaces to closed preordered spaces, not even under convexity, and thus that the closed preorder property is in fact much more interesting since, as we shall see, it allows us to reach better separability properties.

Example 2.1. Let $E = [0,1]^2$ with the usual product topology $\mathscr{T}$. Evidently $(E, \mathscr{T})$ has very good topological properties: it is second countable, Hausdorff, compact and even complete with respect to the Euclidean metric. Let (x, y) be coordinates on E and let $\leq$ be the order defined as follows

$$\begin{aligned} i((x,y)) = \{(x',y') :\ x' = x \text{ and, } & \text{if } x > 0,\ y \leq y'; \\ & \text{if } x = 0 \text{ and } y \leq 1/2,\ y \leq y' \leq 1/2; \\ & \text{if } x = 0 \text{ and } y > 1/2,\ y = y'\}. \end{aligned}$$

With this choice $\leq$ is completely determined, and $(E, \mathscr{T}, \leq)$ can be checked to be a convex semiclosed ordered space which is not a closed ordered space.

We need to state the next two propositions that generalize to preorders two corresponding propositions due to Nachbin [29, Prop. 4,5, Chap. I]. Actually the proofs given by Nachbin for the case of an order work also in this case without any modification. For this reason they are omitted.

Proposition 2.2. *Let E be a closed preordered space. For every compact $K \subset E$, we have $d(K) = D(K)$ and $i(K) = I(K)$, that is, the decreasing and increasing hulls are closed.*

Proposition 2.3. *Let E be a compact closed preordered space. Let $F \subset V$ where F is increasing and V is open, then there is an open increasing set W such that $F \subset W \subset V$. An analogous statement holds in the decreasing case.*

Every compact space equipped with a closed order is normally ordered [29, Theor. 4, Chap. I]. We shall need a slightly stronger statement.

Theorem 2.4. *Every compact space E equipped with a closed preorder is a normally preordered space.*

Proof. If $A \cap B = \emptyset$ with A closed decreasing and B closed increasing, for every $x \in A$ and $y \in B$ we have $d(x) \cap i(y) = \emptyset$, thus there is (see [29, Prop. 1, Chap. I]) a decreasing neighborhood $U(x, y)$ of x and an increasing neighborhood $V(y, x)$

of y (they are not necessarily open) such that $U(x,y)\cap V(y,x)=\emptyset$. Since A and B are closed subsets of a compact set they are compact and $\cup_{y\in B}V(y,x)\supset B$ thus there are points $y_i \in B$, $i=1,\cdots,k$, such that, defined the increasing neighborhood of B, $V(x):=\cup_i V(y_i,x)$, and the decreasing neighborhood of x, $U(x):=\cap_i U(x,y_i)$ we have $U(x)\cap V(x)=\emptyset$. The neighborhoods $U(x)$, $x\in A$ are such that $\cup_{x\in A}U(x)\supset A$ thus we can find x_j, $j=1,\cdots,n$ such that defined the decreasing neighborhood of A, $U':=\cup_j U(x_j)$, and the increasing neighborhood of B, $V':=\cap_j V(x_j)$, we have $U'\cap V'=\emptyset$. Finally, by Prop. 2.3 there are an open decreasing set U such that $A\subset U\subset U'$, and an open increasing set V such that $V'\supset V\supset B$, from which the thesis i.e. $U\cap V=\emptyset$. □

A subset $S\subset E$ with the induced topology $\mathscr{T}_S$ and the induced preorder $\le_S$ is a topological preordered space hence called *subspace*. In general it is not true that every open increasing (decreasing) set on S is the intersection of an open increasing (resp. decreasing) set on E with S. If this is the case S is called a *preordered subspace* [31, 24, 17].

Proposition 2.5. *Every subspace of a (semi)closed preordered space is a (semi)-closed preordered space.*

Proof. Let $(E,\mathscr{T},\le)$ be semiclosed preordered. Given $x\in S$, let $i_S(x)=\{y\in S : x\le_S y\}=\{y\in S: x\le y\}$. From this expression we have $i_S(x)=i(x)\cap S$, thus $i_S(x)$ is closed in the induced topology. Analogously, $d_S(x)$ is closed in the induced topology, that is $(S,\mathscr{T}_S,\le_S)$ is semiclosed preordered.

Let $(E,\mathscr{T},\le)$ be closed preordered, thus $G(\le)$ is closed, and hence the graph of $\le_S$, $G\cap(S\times S)$ is closed in the induced (product) topology $(\mathscr{T}\times\mathscr{T})_{S\times S}$. The equality $(\mathscr{T}\times\mathscr{T})_{S\times S}=(\mathscr{T}_S\times\mathscr{T}_S)$ proves that the graph of $\le_S$ is closed. □

Proposition 2.6. *In a closed preordered space every compact subspace S is a preordered subspace.*

Proof. If $A\subset S$ is closed decreasing then it is compact, thus $d(A)$ is closed decreasing and such that $A=d(A)\cap S$. The proof in the increasing case is analogous. □

Using the previous results it is possible to follow the strategy of the proof that a Hausdorff k_ω-space is a normal space [11] in order to obtain the next theorem.

Theorem 2.7. *Let $(E,\mathscr{T},\le)$ be a k_ω-space endowed with a closed preorder,*[2] *then $(E,\mathscr{T},\le)$ is a normally preordered space.*

We shall give another proof in the next section. It must be noted that Hausdorff locally compact spaces need not be normal thus in theorem 2.7 the k_ω-space condition cannot be weakened to the k-space condition (consider the discrete order).

[2] We shall call these spaces: closed preordered k_ω-spaces. However, this terminology is different from that used in [16] where in their closed ordered k-spaces the term "k-space" includes an additional condition on the upper and lower topologies.

Proof.[3] Let K_n, $K_n \subset K_{n+1}$, $\bigcup_n K_n = E$, be an admissible sequence of compact sets (i.e. that appearing in the definition of k_ω-space). A set O is open if and only if $O \cap K_n$ is open in K_n.

Let A, B be respectively a closed decreasing and a closed increasing set such that $A \cap B = \emptyset$ and denote $A_n = A \cap K_n$, $B_n = B \cap K_n$. By proposition 2.5 and theorem 2.4 every K_n with the induced topology and preorder is a normally preordered space. Let $\tilde{A}_1 = A_1$, $\tilde{B}_1 = B_1$ and let $U_1, V_1 \subset K_1$ be respectively open decreasing and open increasing sets in K_1 (with respect to the induced topology and preorder) such that $D_1(U_1) \cap I_1(V_1) = \emptyset$, $\tilde{A}_1 \subset U_1$, $\tilde{B}_1 \subset V_1$, where D_1, I_1 are the closure operators for the preordered space K_1. They exist, it suffices to apply the preordered normality of the space three times. The set $I_1(V_1)$ being a closed set in K_1 is, regarded as a subset of E, the intersection between a closed set and a compact set thus it is compact. The set $i(I_1(V_1))$ is closed (Prop. 2.2) and analogously, $d(D_1(U_1))$ is closed. Now, let us consider the closed increasing set on K_2 given by $\tilde{B}_2 = [i(I_1(V_1)) \cap K_2] \cup B_2$ and the closed decreasing set given by $\tilde{A}_2 = [d(D_1(U_1)) \cap K_2] \cup A_2$ (see figure 2.7).

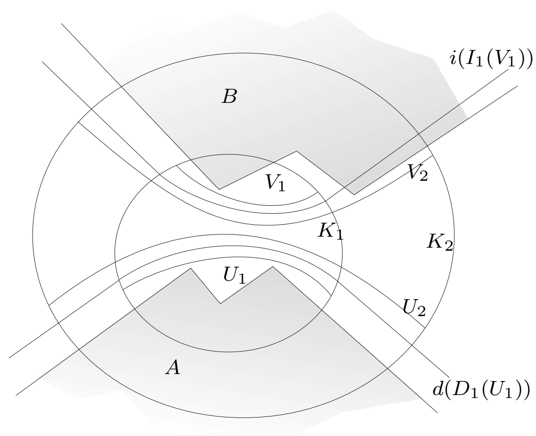

Figure 1: The idea of the proof of theorem 2.7.

They are disjoint because $i(I_1(V_1)) \cap d(D_1(U_1)) = \emptyset$ (otherwise $I_1(V_1) \cap D_1(U_1) \neq \emptyset$, a contradiction), $B_2 \cap A_2 = \emptyset$, and $B_2 \cap d(D_1(U_1)) = \emptyset$ as $i(B_2) \cap D_1(U_1) \subset B_2 \cap D_1(U_1) \subset V_1 \cap D_1(U_1) = \emptyset$. Analogously, $i(I_1(V_1)) \cap A_2 = \emptyset$. Thus, arguing as before we can find $U_2, V_2 \subset K_2$ respectively open decreasing and open increasing sets in K_2 (with respect to the induced topology and preorder) such that $D_2(U_2) \cap I_2(V_2) = \emptyset$, $\tilde{A}_2 \subset U_2$, $\tilde{B}_2 \subset V_2$. Continuing in this way we define at each step $\tilde{B}_{j+1} = [i(I_j(V_j)) \cap K_{j+1}] \cup B_{j+1}$, $\tilde{A}_{j+1} = [d(D_j(U_j)) \cap K_{j+1}] \cup A_{j+1}$. Arguing as before $\tilde{A}_{j+1}, \tilde{B}_{j+1}$ are disjoint closed decreasing and closed increasing subsets of K_{j+1} and since the latter is a normally preordered space there are $U_{j+1}, V_{j+1} \subset K_{j+1}$ respectively open decreas-

[3]This first proof does not appear in the version of the paper published in Order.

ing and open increasing sets in K_{j+1} such that $D_{j+1}(U_{j+1}) \cap I_{j+1}(V_{j+1}) = \emptyset$, $\tilde{A}_{j+1} \subset U_{j+1}$, $\tilde{B}_{j+1} \subset V_{j+1}$.

Note that $V_j \subset \tilde{B}_{j+1} \subset V_{j+1}$ and analogously, $U_j \subset U_{j+1}$. Let $V = \bigcup_j V_j$ and $U = \bigcup_j U_j$. The set V contains B because $B_j \subset \tilde{B}_j \subset V_j$ thus $B = \bigcup_j B_j \subset V$. Analogously, U contains A.

The set V is open because $V \cap K_s = \bigcup_{j \geq 1}(V_j \cap K_s) = \bigcup_{j \geq s}(V_j \cap K_s)$, and the set $V_j \subset K_j$ is open in K_j so that, since for $j \geq s$, $K_s \subset K_j$, $V_j \cap K_s$ is open in K_s and so is the union $V \cap K_s$. The k_ω-space property implies that V is open. Analogously, U is open.

Finally, let us prove that V is increasing. Let $x \in V$ then there is some $j \geq 1$ such that $x \in V_j \subset K_j$. Let $y \in i(x)$, then we can find some $r \geq j$ such that $y \in K_r$. Since $V_j \subset V_r$, $x \in V_r$, and since V_r is increasing on K_r, $y \in V_r$ thus $y \in V$. Analogously, U is decreasing which completes the proof.

□

Corollary 2.8. *Every locally compact σ-compact space equipped with a closed preorder is a normally preordered space.*

We can regard as a corollary the known result [11],

Corollary 2.9. *Every Hausdorff k_ω-space is a normal space.*

Proof. Apply theorem 2.7 to the discrete order and use the fact that the Hausdorff condition is equivalent to the closure of the graph of the discrete order, namely the diagonal $\Delta = \{(x, y) : x = y\}$. □

By a result due to Milnor [26, Lemma 2.1] Hausdorff k_ω-spaces are finitely productive. The locally compact σ-compact spaces are finitely productive even if they are not Hausdorff [5], thus closed preordered locally compact σ-compact spaces are finitely productive. The closed preordered locally compact σ-compact spaces provide our main example because, by using the non-Hausdorff generalization of Morita's theorem, we get the following result.

Corollary 2.10. *Every closed preordered k_ω-space (k-space) E is the quotient of a closed preordered σ-compact (resp. paracompact) locally compact space E', $\pi : E' \to E$, in such a way that $G(\leq') = (\pi \times \pi)^{-1}(G(\leq))$.*

Proof. Obvious from theorem 1.1 because $\leq'$ so defined is a closed preorder. □

3 A proof based on an extension theorem

A function $f : E \to \mathbb{R}$ is *isotone*, if $x \leq y \Rightarrow f(x) \leq f(y)$. Nachbin proved that a continuous isotone function $f : S \to [0, 1]$ defined on a compact subset S of a normally ordered space E can be extended to a function $F : E \to [0, 1]$ on the whole space preserving continuity and the isotone property [29, Theor. 6, Chap. I].[4] Unfortunately, he uses the order condition (and hence the implied Hausdorff

[4]The fact that the theorem holds with the functions f, F, taking values in $[0, 1]$ is evident from Nachbin's proof but is not stated in the original theorem.

condition) and we need therefore to generalize the theorem to the preordered case. Levin gives a similar result for closed preorders on a compact set E [20, Lemma 2], [21] [22, Theor. 6.1], in the context of the mass transfer problem, nevertheless we prefer to give a proof closer in spirit to Nachbin's topological approach.

Theorem 3.1. *Let E be a normally preordered space and let S be a subspace. Let $f : S \to [0,1]$ be continuous and isotone on S. In order that the function f be extendible to a continuous isotone function $F : E \to [0,1]$, it is necessary and sufficient that*

$$\xi, \xi' \in [0,1],\ \xi < \xi' \Rightarrow D(f^{-1}([0,\xi])) \cap I(f^{-1}([\xi',1])) = \emptyset. \tag{1}$$

The proof of this theorem is the same as the proof of [29, Theor. 2]. Indeed, in the body of that proof the image of the original and extended functions is in [0,1] and, rather surprisingly, a close check of the proof (also of the omitted continuity part [15, Chap. 4, Lemma 3]) shows that it does not depend on the closure condition on S which is imposed in the theorem statement.

Remark 3.2. Instead of rechecking Nachbin's proof [29, Theor. 2] one might just follow the argument below to show that f in theorem 3.1 can be extended to $\overline{S}$ preserving continuity, the isotone property and Eq. (1). Then one could apply Nachbin's result [29, Theor. 2] and get theorem 3.1.

Under the hypotheses of Theorem 3.1, let $x \in \overline{S} \backslash S$. Let $(U_\alpha)_\alpha$ be a neighborhood basis of x. Every U_α meets S. Thus $(U_\alpha \cap S)_\alpha$ is a filter basis on S. Let $\eta = \liminf_\alpha f(U_\alpha \cap S)$ and $\eta' = \limsup_\alpha f(U_\alpha \cap S)$. If $\eta < \eta'$ we can find $\eta < \xi < \xi' < \eta'$. By hypothesis $D(f^{-1}([0,\xi])) \cap I(f^{-1}([\xi',1])) = \emptyset$. But x belongs to this intersection as it belongs to $\overline{f^{-1}([0,\xi])} \cap \overline{f^{-1}([\xi',1])}$, which gives a contradiction. Thus $\eta = \eta'$ and f can be extended continuously to x. The extended function $\tilde{f} : \overline{S} \to [0,1]$ is isotone. Indeed, the inequality $\tilde{f}(y) < \tilde{f}(x)$ implies that there are $\zeta < \zeta'$ such that $\tilde{f}(y) < \zeta < \zeta' < \tilde{f}(x)$. By continuity of $\tilde{f}$, $x \in \overline{f^{-1}([\zeta',1])}$ and $y \in \overline{f^{-1}([0,\zeta])}$ and using $D(f^{-1}([0,\zeta])) \cap I(f^{-1}([\zeta',1])) = \emptyset$, we conclude that $x \not\lesssim y$. Finally, let $\xi, \xi' \in [0,1]$, $\xi < \xi'$ and choose $\zeta, \zeta' \in [0,1]$, $\xi < \zeta < \zeta' < \xi'$ then $D(f^{-1}([0,\zeta])) \cap I(f^{-1}([\zeta',1])) = \emptyset$ but $D(\tilde{f}^{-1}([0,\xi])) \subset D(f^{-1}([0,\zeta]))$ and $I(\tilde{f}^{-1}([\xi',1])) \subset I(f^{-1}([\zeta',1]))$ from which it follows $D(\tilde{f}^{-1}([0,\xi])) \cap I(\tilde{f}^{-1}([\xi',1])) = \emptyset$.

This generalization allows us to remove the closure condition in [29, Theor. 3], which therefore reads

Theorem 3.3. *Let E be a normally preordered space and let S be a subspace with the property that if $X, Y \subset S$ satisfy $D_S(X) \cap I_S(Y) = \emptyset$ then $D(X) \cap I(Y) = \emptyset$. Then every continuous isotone function $f : S \to [0,1]$ can be extended to a continuous isotone function $F : E \to [0,1]$.*

If we are in the discrete order case and S is closed, the assumption on theorem 3.3 is satisfied, thus one recovers Tietze's extension theorem for bounded functions. We can now prove (note that S need not be closed; see also remark 4.2 for a different proof).

Theorem 3.4. *Let E be a normally preordered space and let S be a compact subspace, then any continuous isotone function $f : S \to [0, 1]$ can be extended to a continuous isotone function $F : E \to [0, 1]$.*

Proof. Let $X, Y \subset S$ be such that $D_S(X) \cap I_S(Y) = \emptyset$. The set $D_S(X)$ is a closed subset of the compact set S thus it is compact which implies that, by Prop. 2.2, $d(D_S(X))$ is closed. Analogously, $i(I_S(Y))$ is closed. Moreover, $d(D_S(X)) \cap i(I_S(Y)) = \emptyset$ for if $y \in d(D_S(X)) \cap i(I_S(Y))$ there are $z \in D_S(X)$ and $x \in I_S(Y)$, such that $x \leq y \leq z$ which implies $x \leq z$ or $\emptyset \neq i_S(I_S(Y)) \cap D_S(X) = I_S(Y) \cap D_S(X)$, a contradiction. As a consequence, $D(X) \cap I(Y) \subset d(D_S(X)) \cap i(I_S(Y)) = \emptyset$. The desired conclusion follows now from theorem 3.3. □

Lemma 3.5. *(extension and separation lemma) Let $(E, \mathscr{T}, \leq)$ be a closed preordered compact space. Let K be a (possibly empty) compact subset of E, let $A \subset E$ be a closed decreasing set and let $B \subset E$ be a closed increasing set such that $A \cap B = \emptyset$. Let $f : K \to [0, 1]$ be a continuous isotone function on K such that $A \cap K \subset f^{-1}(0)$ and $B \cap K \subset f^{-1}(1)$. Then there is a continuous isotone function $F : E \to [0, 1]$ which extends f such that $A \subset F^{-1}(0)$ and $B \subset F^{-1}(1)$.*

Proof. By theorem 2.4 the topological preordered space E is a normally preordered space. Since A and B are closed and hence compact subsets of E, the set $K' = A \cup K \cup B$ is a compact subset of E. The function $f' : K' \to [0, 1]$ defined by $f'|_A = 0$, $f'|_K = f$, $f'|_B = 1$, is isotone.

Let us prove that f' is continuous on K' with the induced topology. Clearly $f'^{-1}([0, 1])$ is closed in K' as it coincides with K'. We need only to prove that $f'^{-1}([\alpha, 1])$, $\alpha > 0$, is closed in K', the proof for the case $f'^{-1}([0, \beta])$, $\beta < 1$, being analogous. The set $f^{-1}([\alpha, 1])$ being a closed subset of K is a compact subset of E thus $I(f^{-1}([\alpha, 1])) = i(f^{-1}([\alpha, 1]))$. But $A \cap f^{-1}([\alpha, 1]) = \emptyset$ and A is decreasing thus $A \cap I(f^{-1}([\alpha, 1])) = \emptyset$. Since f is continuous on K there is some closed set C in E such that $f^{-1}([\alpha, 1]) = C \cap K$. The closed set $C' = C \cap I(f^{-1}([\alpha, 1]))$ has again the property $f^{-1}([\alpha, 1]) = C' \cap K$ and is disjoint from A. Now we can write

$$f'^{-1}([\alpha, 1]) = B \cup (C' \cap K) = B \cup (C' \cap K'),$$

which proves that $f'^{-1}([\alpha, 1])$ is the union of two closed subsets of K'. We conclude that f' is continuous on K'. By theorem 3.4 f' can be extended to a continuous isotone function $F : E \to [0, 1]$, which is the desired function. □

Theorem 3.6. *(improved extension and separation result) Let $(E, \mathscr{T}, \leq)$ be a k_ω-space equipped with a closed preorder. Let K be a (possibly empty) compact subset, let D be closed decreasing and let I be closed increasing, $D \cap I = \emptyset$. Let $f : K \to [0, 1]$ be a continuous isotone function on K such that $D \cap K \subset f^{-1}(0)$ and $I \cap K \subset f^{-1}(1)$. Then there is a continuous isotone function $F : E \to [0, 1]$ which extends f such that $D \subset F^{-1}(0)$ and $I \subset F^{-1}(1)$.*

Proof. Let K_i, $E = \bigcup_i K_i$, be an admissible sequence according to the definition of k_ω-space. Without loss of generality we can assume $K_i \subset K_{i+1}$ and $K_1 = K$.

By theorem 2.4 and proposition 2.5 each subset K_i endowed with the induced topology and preorder is a normally preordered space. Define $f_1 : K_1 \to [0,1]$, by $f_1 = f$. We make the inductive assumption that there is a continuous isotone function $f_i : K_i \to [0,1]$ such that $D \cap K_i \subset f_i^{-1}(0)$ and $I \cap K_i \subset f_i^{-1}(1)$. Applying lemma 3.5 to $E = K_{i+1}$ with the induced order, $A = D \cap K_{i+1}$, $B = I \cap K_{i+1}$, and compact subspace K_i, we get that there is a continuous isotone function $f_{i+1} : K_{i+1} \to [0,1]$ which extends f_i such that $D \cap K_{i+1} \subset f_{i+1}^{-1}(0)$ and $I \cap K_{i+1} \subset f_{i+1}^{-1}(1)$. We conclude that there is an isotone function $F : E \to [0,1]$ defined by $F|_{K_i} = f_i$ such that $D \subset F^{-1}(0)$ and $I \subset F^{-1}(1)$.

We recall that in a k_ω-space with admissible sequence K_i, $K_i \subset K_{i+1}$, a function $g : E \to \mathbb{R}$ is continuous if and only if for every i, $g|_{K_i}$ is continuous in the subspace K_i. By construction, the function f_i is continuous in K_i thus F is continuous. □

We can give a second proof to theorem 2.7.

Second Proof. Let $D \subset E$ be a closed decreasing subset and let $I \subset E$ be a closed increasing subset such that $D \cap I = \emptyset$. Let $K = \emptyset$, then by theorem 3.6 there is a continuous isotone function $F : E \to [0,1]$ such that $D \subset F^{-1}(0)$ and $I \subset F^{-1}(1)$. The open sets $\{x : F(x) > 1/2\}$ and $\{x : F(x) < 1/2\}$, are respectively open increasing, open decreasing, disjoint and containing respectively I and D, thus $(E, \mathscr{T}, \le)$ is a normally preordered space. □

Example 3.7. We give an example of normally preordered space which admits a non-closed subset S, which satisfies the assumptions of theorem 3.3. Let $E = \mathbb{R} \times S^1$, $S^1 = [0, 2\pi)$, be equipped with the product preorder $\le$, where $\mathbb{R}$ is endowed with the usual order $\preceq$, and S^1 is given the indiscrete preorder. Let the topology $\mathscr{T}$ on E be the coarsest topology which makes the projection on the first factor, $\pi : E \to \mathbb{R}$, continuous. By theorem 2.7 E is normally preordered (or use the remark 4.1 below, and the fact that $(\mathbb{R}, \preceq)$ is normally ordered). The subset $S = [0, \pi]^2$ is compact, thus it satisfies the assumptions of theorem 3.3, but non-closed, its closure being $\overline{S} = [0, \pi] \times S^1$. The subset $S = (0, \pi)^2$ is non-closed and non-compact but it still satisfies the assumptions of theorem 3.3.

4 The ordered quotient space

Let us introduce the equivalence relation $x \sim y$ on E, given by "$x \le y$ and $y \le x$". Let $E/\!\sim$ be the quotient space, $\mathscr{T}/\!\sim$ the quotient topology, and let $\lesssim$ be defined by, $[x] \lesssim [y]$ if $x \le y$ for some representatives (with some abuse of notation we shall denote with $[x]$ both a subset of E and a point on $E/\!\sim$). The quotient preorder is by construction an order. The triple $(E/\!\sim, \mathscr{T}/\!\sim, \lesssim)$ is a topological ordered space and $\pi : E \to E/\!\sim$ is the continuous quotient projection.

Remark 4.1. Taking into account the definition of quotient topology we have that every open (closed) increasing (decreasing) set on E projects to an open (resp. closed) increasing (resp. decreasing) set on $E/\sim$ and all the latter sets can be regarded as such projections. As a consequence, $(E, \mathscr{T}, \le)$ is a normally preordered space (semiclosed preordered space, regularly preordered space) if and only if $(E/\sim, \mathscr{T}/\sim, \lesssim)$ is a normally ordered space (resp. semiclosed ordered space, regularly ordered space).

Remark 4.2. An alternative proof of theorem 3.4 uses the fact that $E/\sim$ is normally ordered and $\pi(S)$ is compact, so that f can be passed to the quotient, extended using [29, Theor. 6, Chap. I] and then lifted to E.

The closed preordered property does not pass smoothly to the quotient, but in the compact case and in the k_ω-space case, using theorem 2.7 (2.4) and theorem 1.1 we obtain.

Corollary 4.3. *If E is a closed preordered compact space then $E/\sim$ is a closed ordered compact space. If E is a closed preordered k_ω-space then $E/\sim$ is a closed ordered k_ω-space.*

Proof. The first statement is a trivial consequence of theorem 2.4. As for the second statement, by Theor. 1.1 $(E/\sim, \mathscr{T}/\sim)$ is a k_ω-space. Since E is a closed preordered k_ω-space then it is normally preordered from which it follows that $E/\sim$ is a normally ordered space and hence a closed ordered space. □

Remark 4.4. The first statement in the previous result is contained in [7, Lemma 1] but the proof is incorrect again for the tricky Hausdorff condition which they inadvertently use in the quotient. Indeed, they argument as follows: they take a net $([a_\alpha], [b_\alpha]) \in G(\lesssim)$ converging to $([a], [b])$ and prove that a subnet $(a_\beta, b_\beta) \in G(\le)$ converges to some pair $(a', b') \in G(\le)$. Since π is continuous $([a_\beta], [b_\beta])$ converges to $([a'], [b']) \in G(\lesssim)$ (and also to $([a], [b])$) but this does not mean that $([a'], [b']) = ([a], [b])$ as the uniqueness of the limit requires the Hausdorff condition and this is assured only *after* it is proved that $E/\sim$ is a closed ordered space.

Remark 4.5. In the Hausdorff case, the second statement in corollary 4.3 can be proved using the strategy contained in [19]. If E is a Hausdorff k_ω-space then $E/\sim$ is a Hausdorff k_ω-space [19, Prop. 2.3b], then $\pi \times \pi$ is a quotient map [19, Prop. 2.3a, 2.2] which implies since $G(\le) = (\pi \times \pi)^{-1}(G(\lesssim))$ that $G(\lesssim)$ is closed. One can then work out the proof of theorem 2.7 in the ordered framework of $E/\sim$ using remark 4.1.

5 The existence of continuous utilities

Let us write $x < y$ if $x \le y$ and $y \not\le x$. A *utility* is a function $f : E \to \mathbb{R}$ such that "$x \sim y \Rightarrow f(x) = f(y)$ and $x < y \Rightarrow f(x) < f(y)$". We say that the preorder admits a *representation* by a family of functions $\mathcal{F}$ if "$x \le y$ iff $\forall f \in \mathcal{F}, f(x) \le f(y)$". It is easy to prove that the preorder of a normally preordered

space is represented by the family of continuous isotone functions [29, Theor. 1]. It is interesting to investigate under which conditions a continuous utility or, more strongly, a representation through continuous utilities exists. This problem has been thoroughly investigated, especially in the economics literature. The simplest approach passes through the assumption that the preordered space under consideration is normally preordered [25], although alternative strategies based on weaker hypothesis have also been investigated [13]. The representation of relations through isotone and utility functions is still an active field of research [30, 3, 4].

The reader must be warned that some authors call *utility* what we call *isotone* function [14, 3, 9] and use the word *representation* for the existence of just one utility, although one utility does not allow us to recover the preorder. This unfortunate circumstance comes from the fact that in economics most terminology was introduced in connection with the total preorder case, that is, before the importance of the general preorder case was recognized.

Let us recall that a *perfectly normally preordered* space is a semiclosed preordered space such that if A is a closed decreasing set and B is a closed increasing set with $A \cap B = \emptyset$ then there is a continuous isotone function $f : E \to [0, 1]$ such that $A = f^{-1}(0)$ and $B = f^{-1}(1)$. Clearly, perfectly normally preordered spaces are normally preordered spaces.

A closed decreasing (increasing) set S is *functionally-preordered closed* if there is a continuous isotone function $f : E \to [0, 1]$ such that $S = f^{-1}(0)$ (resp. $S = f^{-1}(1)$). They can also be called decreasing (increasing) *zero sets* as done in [23]. A pair (A, B), $A \cap B = \emptyset$, is *functionally-preordered closed* if there is a continuous isotone function $f : E \to [0, 1]$ such that $A = f^{-1}(0)$ and $B = f^{-1}(1)$. The next result is stated without proof in [23].

Proposition 5.1. *If in the pair (A, B), $A \cap B = \emptyset$, with A closed decreasing and B closed increasing, A is functionally-preordered closed and B is functionally-preordered closed then (A, B) is functionally-preordered closed.*

Proof. Let $A = g^{-1}(0)$ and $B = h^{-1}(1)$ with $g, h : E \to [0, 1]$ continuous isotone functions. Let $\alpha : ([0, 1] \times [0, 1])\backslash(0, 1) \to [0, 1]$ be a continuous function which is isotone according to the product order on the square. Let α be also such that $\alpha^{-1}(0) = \{(0, y) : y \in [0, 1)\}$ and $\alpha^{-1}(1) = \{((x, 1) : x \in (0, 1]\}$. A possible example is $\alpha = \frac{1+y}{2} x^{(1-y)/2}$; another example is $\alpha = 1/[1 + (1 - y)/x]$. The function $f = \alpha(g, h)$ is isotone, continuous, and satisfies $f^{-1}(0) = g^{-1}(0) = A$, $f^{-1}(1) = h^{-1}(0) = B$. □

The next result extends a known result for topological spaces [32, Theor. 16.8].

Proposition 5.2. *Every regularly preordered Lindelöf space is a normally preordered space.*

Proof. Let A and B be closed disjoint sets which are respectively decreasing and increasing. Since $A \cap B = \emptyset$, by preorder regularity for each $x \in A$ there is an open decreasing set $U_x \ni x$ such that $D(U_x) \cap B = \emptyset$. The collection $\{U_x\}$

covers A and since the Lindelöf property is hereditary with respect to closed subspaces, there is a countable subcollection $\{U_i\}$ with the same property. In the same way we find a countable collection of open increasing sets $\{V_i\}$ which covers B and such that $I(V_i) \cap A = \emptyset$. Let us define the sequence of open decreasing sets $W_1 = U_1$, $W_{n+1} = U_{n+1} \backslash [\cup_{i=1}^n I(V_i)]$, and the sequence of open increasing sets $E_1 = V_1 \backslash D(U_1)$, $E_n = V_n \backslash [\cup_{i=1}^n D(U_i)]$. The open disjoint sets $U' = \cup_{n=1}^\infty W_n$ and $V' = \cup_{n=1}^\infty E_n$ are respectively decreasing and increasing and contain respectively A and B. □

Theorem 5.3. *Every second countable regularly preordered space $(E, \mathscr{T}, \le)$ is a perfectly normally preordered space.*

Proof. As second countability implies the Lindelöf property, by proposition 5.2 E is normally preordered. Let A be a closed decreasing set, we have only to prove that it is functionally-preordered closed, the proof in the closed increasing case being analogous. Suppose A is open then $E \backslash A$ is open, closed and increasing. Setting for $x \in A$, $f(x) = 0$ and for $x \in E \backslash A$, $f(x) = 1$ we have finished. Therefore, we can assume that A is not open and hence that $A \neq E$.

Let $\mathcal{U}$ be a countable base of the topology of E. Let $\mathcal{C} \subset \mathcal{U}$ be the subset whose elements, denoted U_i, $i \ge 1$, are such that $A \cap I(U_i) = \emptyset$. For every $U_k \in \mathcal{C}$, denote with $f_k : E \to [0,1]$ an isotone continuous function which separates A and $I(U_k)$, that is, such that $f_k^{-1}(1) \supset I(U_k)$ and $f_k^{-1}(0) \supset A$ (see [29, Theor. 1]).

Let $x \in E \backslash A$. Applying the preordered normality of the space we can find $\hat{U}$, open decreasing sets such that $A \subset \hat{U} \subset D(\hat{U}) \subset E \backslash i(x)$. As $E \backslash D(\hat{U})$ is open there is some $O \in \mathcal{U}$ such that $x \in O \subset E \backslash D(\hat{U})$. Furthermore, since $E \backslash \hat{U} \supset O$ is closed increasing, $I(O) \subset E \backslash \hat{U}$ which implies $A \cap I(O) = \emptyset$ and hence that there is some $U_k \in \mathcal{C}$ such that $O = U_k$. Thus there is a continuous isotone function f_k such that $f_k^{-1}(1) \supset I(U_k)$ and $f_k^{-1}(0) \supset A$. In other words we have proved that for each $x \in E \backslash A$ there is some k such that $f_k(x) = 1$. Let us consider the function

$$f = \sum_{k=1}^{\infty} \frac{1}{2^k} f_k.$$

This function is clearly isotone, takes values in $[0,1]$ and it is continuous because the series converges uniformly. Since for every k, $f_k^{-1}(0) \supset A$, the same is true for f. Moreover, note that if $x \in E \backslash A$ then there is some $k \ge 1$ such that $f_k(x) > 0$, which implies that $f(x) > 0$ hence $f^{-1}(0) = A$. □

Lemma 5.4. *If a topological preordered space admits a countable continuous isotone function representation, namely if there are continuous isotone functions $g_k : E \to [0,1]$, $k \ge 1$ such that $x \le y \Leftrightarrow \forall k \ge 1, g_k(x) \le g_k(y)$, then there is a countable continuous utility representation, namely there are continuous utility functions $f_k : E \to [0,1]$, $k \ge 1$ such that $x \le y \Leftrightarrow \forall k \ge 1, f_k(x) \le f_k(y)$.*

Proof. Let us consider the function

$$g = \sum_{k=1}^{\infty} \frac{1}{2^k} g_k.$$

This function is clearly isotone, takes values in $[0, 1]$ and it is continuous because the series converges uniformly. Furthermore, if $x \le y$ and $y \not\le x$, then $g(x) < g(y)$ because all the g_k are isotone and there is some $g_{\bar{k}}$ for which $g_{\bar{k}}(x) < g_{\bar{k}}(y)$. Indeed, if it were not so then for every k, $g_k(y) \le g_k(x)$ which implies $y \le x$, a contradiction. We conclude that g is a continuous utility function. Now, define for all $k, n \ge 1$, $\tilde{g}_{kn} = (1 - 1/n)g_k + g/n$, so that $\tilde{g}_{kn}$ is a continuous utility function. We have only to prove that if $x \not\le y$ there are some k, n, such that $\tilde{g}_{kn}(x) > \tilde{g}_{kn}(y)$ but we know that there is a $\tilde{k}$ such that $g_{\tilde{k}}(x) > g_{\tilde{k}}(y)$ (otherwise for every k, $g_k(x) \le g_k(y)$ which implies $x \le y$, a contradiction). Thus $\tilde{g}_{\tilde{k}n}(x) = (1 - 1/n)g_{\tilde{k}}(x) + g(x)/n = \tilde{g}_{\tilde{k}n}(y) + (1 - 1/n)(g_{\tilde{k}}(x) - g_{\tilde{k}}(y)) + (g(x) - g(y))/n$ and thus the desired inequality holds for sufficiently large n. □

Theorem 5.5. *Every second countable regularly preordered space E admits a countable continuous utility representation, that is, there is a countable set $\{f_k, k \ge 1\}$ of continuous utility functions $f_k : E \to [0, 1]$ such that*

$$x \le y \Leftrightarrow \forall k \ge 1, f_k(x) \le f_k(y).$$

Proof. Let $\mathcal{G}$ be the set of continuous isotone functions and for every $g \in \mathcal{G}$ let $G_g = \{(x, y) \in E \times E : g(x) \le g(y)\}$. The set G_g is closed because of the continuity of g. We already know that the preordered space $(E, \mathscr{T}, \le)$ is a normally preordered space (Theor. 5.3), thus $\le$ is represented by $\mathcal{G}$ (by [29, Theor. 1]), namely $G(\le) = \bigcap_{g \in \mathcal{G}} G_g$. As E is second countable, $E \times E$ is second countable and hence hereditary Lindelöf. As a consequence, the intersection of an arbitrary family of closed sets can be written as the intersection of a countable subfamily $\mathcal{G}' \subset \mathcal{G}$. The desired conclusion follows now from lemma 5.4. □

6 Conclusions

Every Hausdorff locally compact space is a completely regular space [32, Theor. 19.3] and hence a regular space, and every second countable regular space (T_3-space) is metrizable (Urysohn's theorem). These results allow us to improve the separability properties of the space. Unfortunately, they have no straightforward analogs in the general preordered case where a quasi-uniformizable space need not be a regularly preordered space [18, Example 1]. One still would like to have some result which improves the preorder separability properties of the space, given suitable compactness or countability conditions on the topology. We have then followed a different path proving that closed preordered k_ω-spaces are normally preordered. We have also proved that second countable regularly preordered spaces are perfectly normally preordered and that they admit a countable continuous utility representation of the preorder.

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References

[1] Akin, E.: *The general topology of dynamical systems.* Providence: Amer. Math. Soc. (1993)

[2] Aumann, R. J.: Utility theory without the completeness axiom. Econometrica **30**, 445–462 (1962)

[3] Bosi, G. and Herden, G.: On a possible continuous analogue of the Szpilrajn theorem and its strengthening by Dushnik and Miller. Order **23**, 271–296 (2006)

[4] Bosi, G. and Isler, R.: *Continuous utility functions for nontotal preorders: A review of recent results*, Springer-Verlag, vol. Preferences and decisions 257 of *Studies in fuzziness and soft computing*, pages 1–10 (2010)

[5] Bourbaki, N.: *Elements of Mathematics: General topology I.* Reading: Addison-Wesley Publishing (1966)

[6] Bridges, D. S. and Mehta, G. B.: *Representations of preference orderings*, vol. 442 of *Lectures Notes in Economics and Mathematical Systems.* Berlin: Springer-Verlag (1995)

[7] Candeal-Haro, J. C., Induráin, E., and Mehta, G. B.: Some utility theorems on inductive limits of preordered topological spaces. Bull. Austral. Math. Soc. **52**, 235–246 (1995)

[8] Engelking, R.: *General Topology.* Berlin: Helderman Verlag (1989)

[9] Evren, O. and Ok, E. A.: On the multi-utility representation of preference relations (June 2008)

[10] Fletcher, P. and Lindgren, W.: *Quasi-uniform spaces*, vol. 77 of *Lect. Notes in Pure and Appl. Math.* New York: Marcel Dekker, Inc. (1982)

[11] Franklin, S. T. and Smith Thomas, B. V.: A survey of k_ω-spaces. Topology Proceedings **2**, 111–124 (1977)

[12] Gierz, G., Hofmann, K. H., Keimel, K., Lawson, J. D., Mislove, M. W., and Scott, D. S.: *Continuous lattices and domains.* Cambridge University Press (2003)

[13] Herden, G.: On the existence of utility functions. Mathematical Social Sciences **17**, 297–313 (1989)

[14] Herden, G. and Pallack, A.: On the continuous analogue of the Szpilrajn theorem I. Mathematical Social Sciences **43**, 115–134 (2002)

[15] Kelley, J. L.: *General Topology.* New York: Springer-Verlag (1955)

[16] Kopperman, R. and Lawson, J.: Bitopological and topological ordered k-spaces. Topology and its Applications **146-147**, 385–396 (2005)

[17] Künzi, H.-P. A.: Completely regular ordered spaces. Order **7**, 283–293 (1990)

[18] Künzi, H.-P. A. and Watson, S.: A metrizable completely regular ordered space. Comment. Math. Univ. Carolinae **35**, 773–778 (1994)

[19] Lawson, J. and Madison, B.: On congruences of cones. Math. Z. **120**, 18–24 (1971)

[20] Levin, V. L.: A continuous utility theorem for closed preorders on a σ-compact metrizable space. Soviet Math. Dokl. **28**, 715–718 (1983)

[21] Levin, V. L.: Measurable utility theorem for closed and lexicographic preference relations. Soviet Math. Dokl. **27**, 639–643 (1983)

[22] Levin, V. L. and Milyutin, A. A.: The problem of mass transfer with a discontinuous cost function and a mass statement of the duality problem for convex extremal problems. Russ. Math. Surv. **34**, 1–78 (1979)

[23] McCallion, T.: Compactifications of ordered topological spaces. Proc. Camb. Phil. Soc. **71**, 463–473 (1972)

[24] McCartan: Separation axioms for topological ordered spaces. Proc. Camb. Phil. Soc. **64**, 965–973 (1968)

[25] Mehta, G.: Some general theorems on the existence of order-preserving functions. Mathematical Social Sciences **15**, 135–143 (1988)

[26] Milnor, J.: Construction of universal bundles, I. Ann. Math. **63**, 272–284 (1956)

[27] Minguzzi, E.: Time functions as utilities. Commun. Math. Phys. **298**, 855–868 (2010)

[28] Morita, K.: On the decomposition spaces of locally compact spaces. Proc. Japan Acad. **32**, 544–548 (1956)

[29] Nachbin, L.: *Topology and order.* Princeton: D. Van Nostrand Company, Inc. (1965)

[30] Ok, E. A.: Utility representation of an incomplete preference relation. J. Econ. Theory **104**, 429–449 (2002)

[31] Priestley, H. A.: Ordered topological spaces and the representation of distributive lattices. Proc. London Math. Soc. **24**, 507–530 (1972)

[32] Willard, S.: *General topology.* Reading: Addison-Wesley Publishing Company (1970)